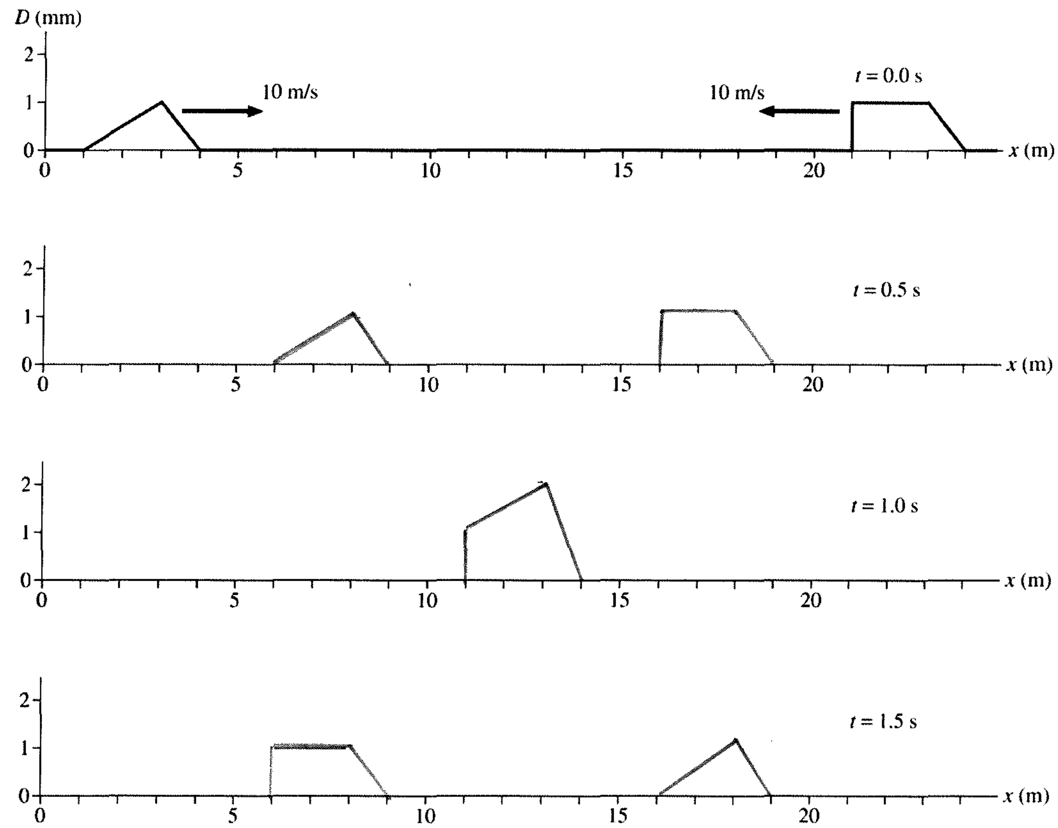
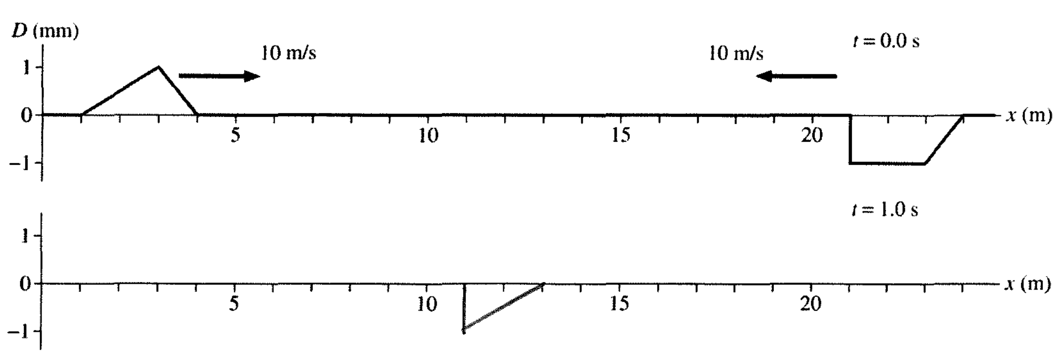
17 Superposition

17.1 The Principle of Superposition

1. Two pulses on a string, both traveling at 10 m/s, are approaching each other. Draw snapshot graphs of the string at the three times indicated.



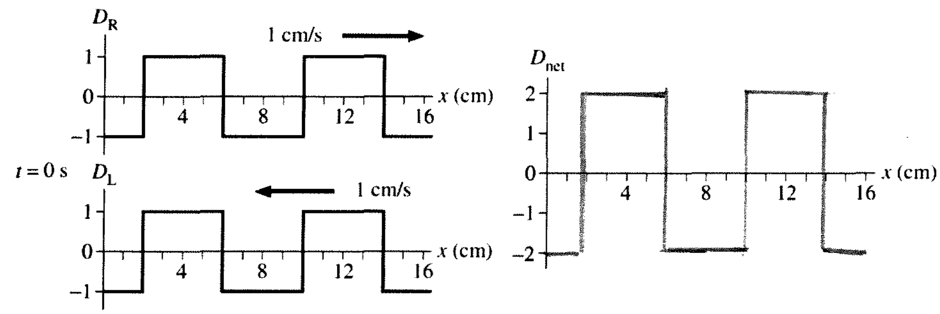
2. Two pulses on a string, both traveling at 10 m/s, are approaching each other. Draw a snapshot graph of the string at $t = 1$ s.

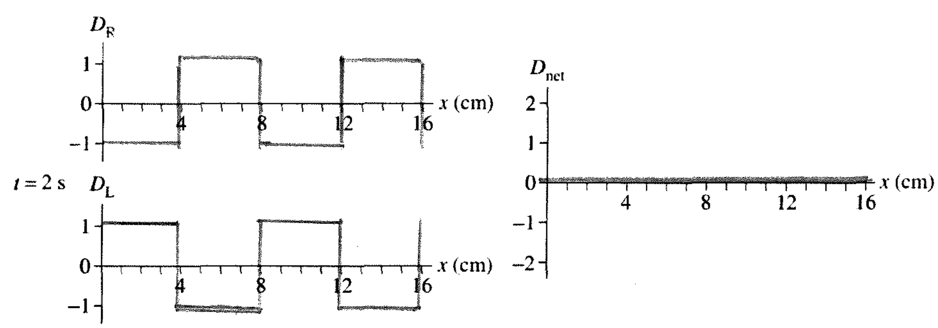


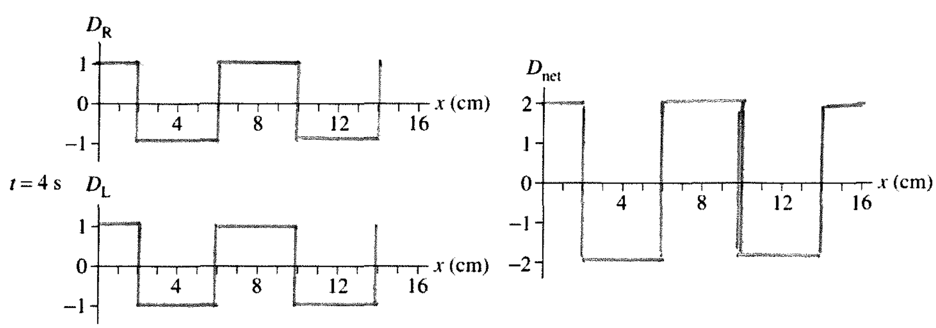
17.2 Standing Waves

17.3 Standing Waves on a String

3. Two waves are traveling in opposite directions along a string. Each has a speed of 1 cm/s and an amplitude of 1 cm. The first set of graphs below shows each wave at $t = 0$ s.
   a. On the axes at the right, draw the superposition of these two waves at $t = 0$ s.
   b. On the axes at the left, draw each of the two displacements every 2 s until $t = 8$ s. The waves extend beyond the graph edges, so new pieces of the wave will move in.
   c. On the axes at the right, draw the superposition of the two waves at the same instant.







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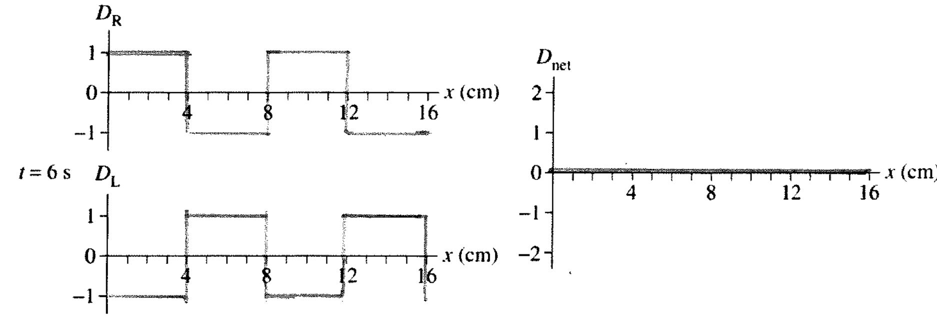
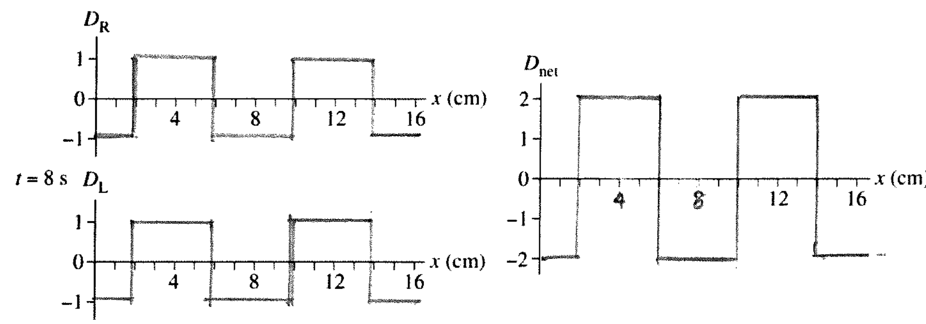
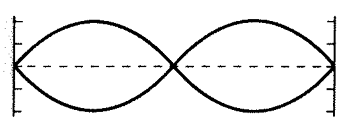
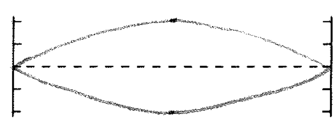




4. The figure shows a standing wave on a string.
   a. Draw the standing wave if the tension is quadrupled while the frequency is held constant.



Original wave, tension T

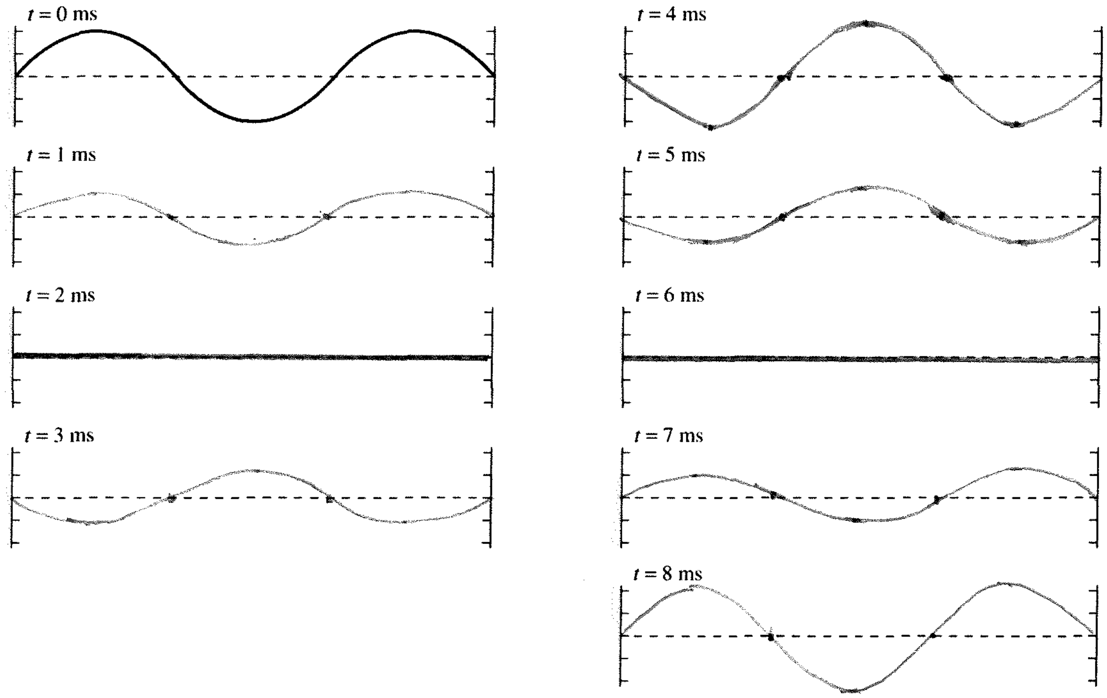


Tension $4T$

   b. Suppose the tension is merely doubled while the frequency remains constant. Will there be a standing wave? If so, how many antinodes will it have? If not, why not?

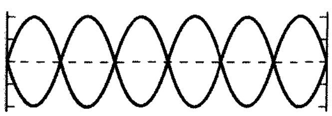
There will be no standing wave. Boundary conditions for standing waves on a string require nodes at both ends: $\lambda_m = 2L/m$ where $m = 1, 2, 3, \ldots$
Originally $\lambda = \frac{v}{f} = \frac{1}{f}\sqrt{\frac{T}{\mu}} = L$ so $m = 2$.
Quadrupling T gives $\lambda = 2L$ so $m = 1$, but doubling T gives $\lambda = \sqrt{2}\,L$ so an integer value for m is not possible, and the condition that requires nodes at both ends cannot be met.

5. This standing wave has a period of 8 ms. Draw snapshot graphs of the string every 1 ms from $t = 1$ ms to $t = 8$ ms. Think carefully about the proper amplitude at each instant.

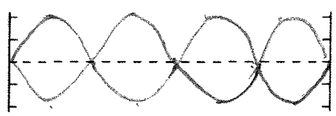


6. The figure shows a standing wave on a string. It has frequency f.

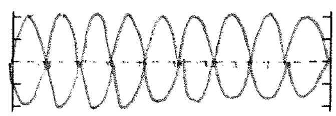
a. Draw the standing wave if the frequency is changed to $\frac{2}{3}f$ and to $\frac{3}{2}f$.



Original wave, frequency f



Frequency $\frac{2}{3}f$



Frequency $\frac{3}{2}f$

b. Is there a standing wave if the frequency is changed to $\frac{1}{4}f$? If so, how many antinodes does it have? If not, why not?

There will be no standing wave. Boundary conditions for standing waves on a string require nodes at both ends:

$\lambda_m = \frac{2L}{m}$ where $m = 1, 2, 3 \ldots$

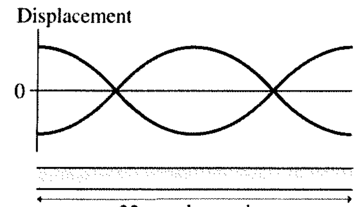
Originally $\lambda = \frac{v}{f} = \frac{2L}{6}$ so $m = 6$

For $\frac{2}{3}f$, $\lambda = \frac{v}{\frac{2}{3}f} = \frac{3}{2}\frac{v}{f} = \frac{3}{2}\left(\frac{2L}{6}\right) = \frac{2L}{4}$ so $m = 4$

For $\frac{3}{2}f$, $\lambda = \frac{v}{\frac{3}{2}f} = \frac{2}{3}\frac{v}{f} = \frac{2}{3}\left(\frac{2L}{6}\right) = \frac{2L}{9}$ so $m = 9$

For $\frac{1}{4}f$, $\lambda = \frac{v}{f/4} = 4\frac{v}{f} = 4\left(\frac{2L}{6}\right) = \frac{2L}{6/4} = \frac{2L}{3/2}$ does not give integer value for m, so the condition that requires nodes at both ends cannot be met.

17.4 Standing Sound Waves and Musical Acoustics



7. The picture shows a displacement standing sound wave in a 32-mm-long tube of air that is open at both ends.
   a. Which mode (value of m) standing wave is this? 2
   b. Are the air molecules vibrating vertically or horizontally? Explain.

   Horizontally. Sound waves are longitudinal.

   c. At what distances from the left end of the tube do the molecules oscillate with maximum amplitude?

   0 mm, 16 mm, 32 mm
   (at the antinodes)

8. The purpose of this exercise is to visualize the motion of the air molecules for the standing wave of Exercise 7. On the next page are nine graphs, every one-eighth of a period from $t = 0$ to $t = T$. Each graph represents the displacements at that instant of time of the molecules in a 32-mm-long tube. Positive values are displacements to the right; negative values are displacements to the left.
   a. Consider nine air molecules that, in equilibrium, are 4 mm apart and lie along the axis of the tube. The top picture on the right shows these molecules in their equilibrium positions. The dotted lines down the page—spaced 4 mm apart—are reference lines showing the equilibrium positions. Read each graph carefully, then draw nine dots to show the positions of the nine air molecules at each instant of time. The first one, for $t = 0$, has already been done to illustrate the procedure.

   Note: It's a good approximation to assume that the left dot moves in the pattern 4, 3, 0, −3, −4, −3, 0, 3, 4 mm; the second dot in the pattern 3, 2, 0, −2, −3, −2, 0, 2, 3 mm; and so on.

   b. At what times does the air reach maximum compression, and where does it occur?

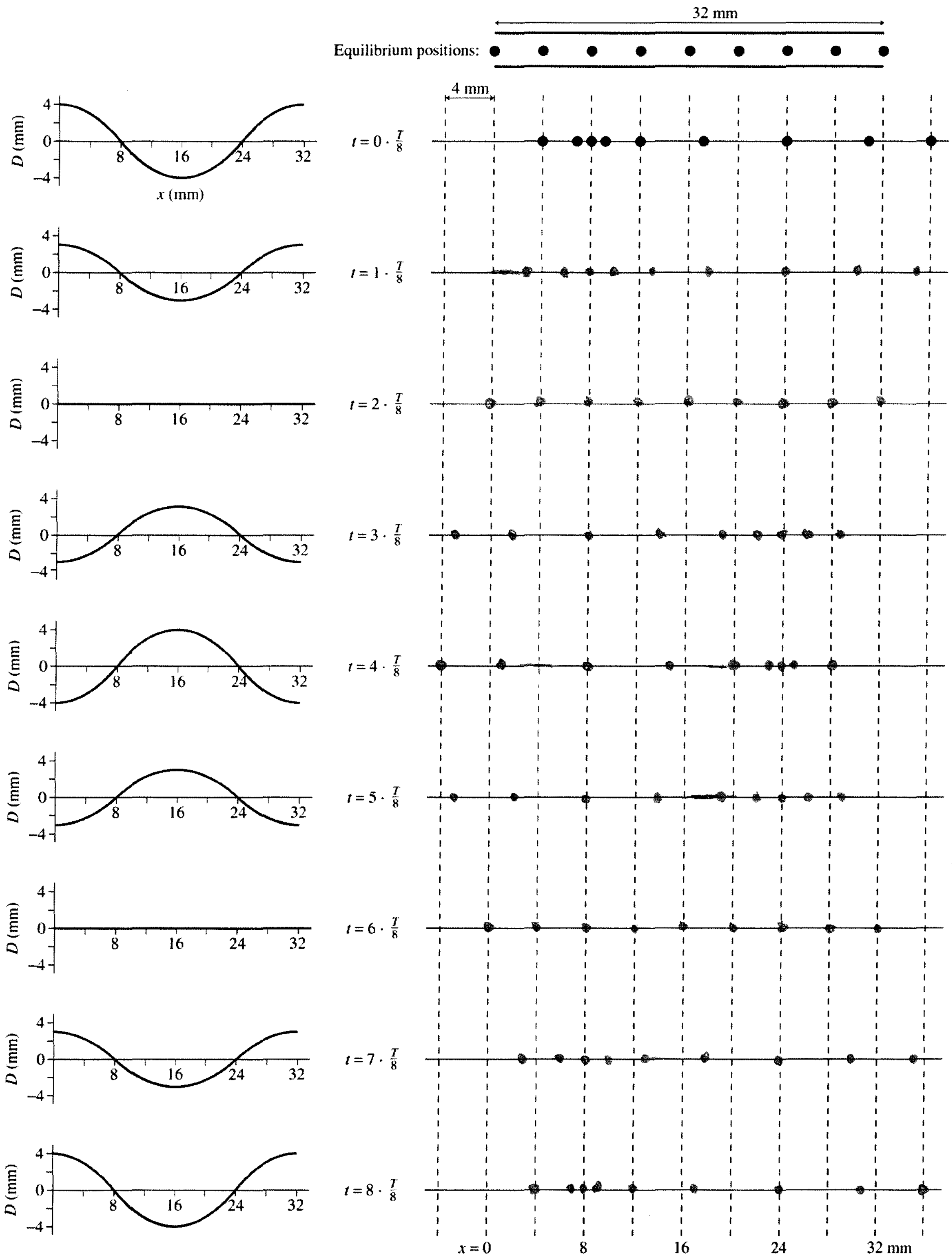
   Max compression at time 0 Max compression at position 8 mm

   $4\left(\frac{T}{8}\right) = \frac{T}{2}$ 24 mm

   $8\left(\frac{T}{8}\right) = T$ 8 mm

   c. What is the relationship between the positions of maximum compression and the nodes of the standing wave?

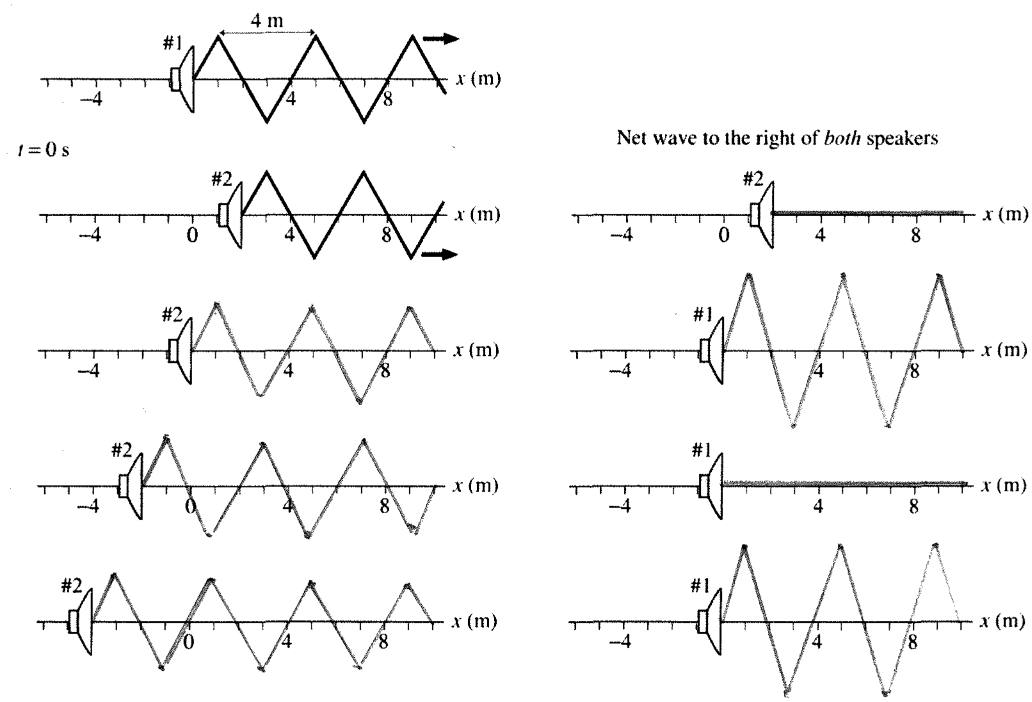
   The points of maximum compression are nodes.


32 mm
Equilibrium positions:
4 mm
$t = 0 \cdot \frac{T}{8}$
$t = 1 \cdot \frac{T}{8}$
$t = 2 \cdot \frac{T}{8}$
$t = 3 \cdot \frac{T}{8}$
$t = 4 \cdot \frac{T}{8}$
$t = 5 \cdot \frac{T}{8}$
$t = 6 \cdot \frac{T}{8}$
$t = 7 \cdot \frac{T}{8}$
$t = 8 \cdot \frac{T}{8}$
$x = 0$
8
16
24
32 mm
D (mm)
x (mm)

17.5 Interference in One Dimension

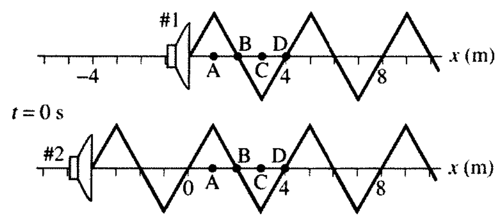
17.6 The Mathematics of Interference

9. The figure shows a snapshot graph at $t = 0$ s of loudspeakers emitting triangular-shaped sound waves. Speaker 2 can be moved forward or backward along the axis. Both speakers vibrate in phase at the same frequency. The second speaker is drawn below the first, so that the figure is clear, but you want to think of the two waves as overlapped as they travel along the x-axis.
   a. On the left set of axes, draw the $t = 0$ s snapshot graph of the second wave if speaker 2 is placed at each of the positions shown. The first graph, with $x_{speaker} = 2$ m, is already drawn.



   b. On the right set of axes, draw the superposition $D_{net} = D_1 + D_2$ of the waves from the two speakers. D_{net} exists only to the right of *both* speakers. It is the net wave traveling to the right.
   c. What separations between the speakers give constructive interference? 0, 4 m m
   d. What are the $\Delta x/\lambda$ ratios at the points of constructive interference? 0, 1
   e. What separations between the speakers give destructive interference? + 2 m, − 2 m
   f. What are the $\Delta x/\lambda$ ratios at the points of destructive interference? ± 1/2

10. Two loudspeakers are shown at $t = 0$ s. Speaker 2 is 4 m behind speaker 1.



a. What is the wavelength λ? 4 m

b. Is the interference constructive or destructive?

Constructive

c. What is the phase constant ϕ_{10} for wave 1? 0

What is the phase constant ϕ_{20} for wave 2? 0

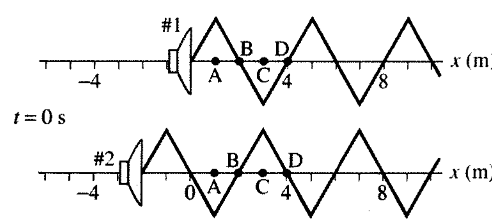
d. At points A, B, C, and D on the x-axis, what are:

- The distances x_1 and x_2 to the two speakers?
- The path length difference $\Delta x = x_2 - x_1$?
- The phases ϕ_1 and ϕ_2 of the two waves at the point (not the phase constant)?
- The phase difference $\Delta\phi = \phi_2 - \phi_1$?

Point A is already filled in to illustrate.

	x_1	x_2	Δx	ϕ_1	ϕ_2	$\Delta\phi$
Point A	1 m	5 m	4 m	0.5π rad	2.5π rad	2π rad
Point B	2 m	6 m	4 m	π rad	3π rad	2π rad
Point C	3 m	7 m	4 m	1.5π rad	3.5π rad	2π rad
Point D	4 m	8 m	4 m	2π rad	4π rad	2π rad

e. Now speaker 2 is placed 2 m behind speaker 1. Is the interference constructive or destructive?



Destructive

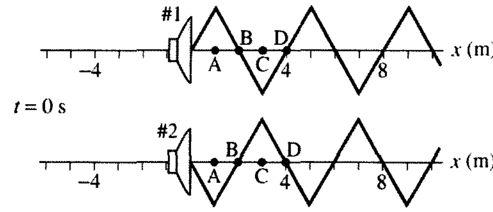
f. Repeat step d for the points A, B, C, and D.

	x_1	x_2	Δx	ϕ_1	ϕ_2	$\Delta\phi$
Point A	1 m	3 m	2 m	0.5π rad	1.5π rad	π rad
Point B	2 m	4 m	2 m	π rad	2π rad	π rad
Point C	3 m	5 m	2 m	1.5π rad	2.5π rad	π rad
Point D	4 m	6 m	2 m	2π rad	3π rad	π rad

g. When the interference is constructive, what is $\Delta x/\lambda$? 1 What is $\Delta\phi$? 2π

h. When the interference is destructive, what is $\Delta x/\lambda$? 1/2 What is $\Delta\phi$? π

11. Two speakers are placed side-by-side at $x = 0$ m. The waves are shown at $t = 0$ s.



a. Is the interference constructive or destructive?

b. What is the phase constant ϕ_{10} for wave 1? 0

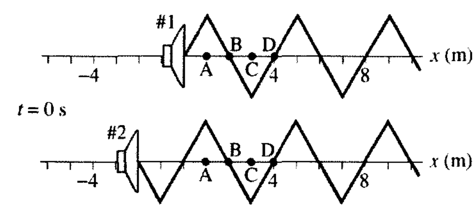
What is the phase constant ϕ_{10} for wave 2? π

c. At points A, B, C, and D on the x-axis, what are:

- The distances x_1 and x_2 to the two speakers?
- The path length difference $\Delta x = x_2 - x_1$?
- The phases ϕ_1 and ϕ_2 of the two waves at the point (not the phase constant)?
- The phase difference $\Delta\phi = \phi_2 - \phi_1$?

	x_1	x_2	Δx	ϕ_1	ϕ_2	$\Delta\phi$
Point A	1 m	1 m	0 m	0.5π	1.5π	π
Point B	2 m	2 m	0 m	π	2π	π
Point C	3 m	3 m	0 m	1.5π	2.5π	π
Point D	4 m	4 m	0 m	2π	3π	π

d. Speaker 2 is moved back 2 m. Does this change its phase constant ϕ_0?



No. Initial phase is at the source.

e. Is the interference constructive or destructive?

Constructive.

f. Repeat step c for the points A, B, C, and D.

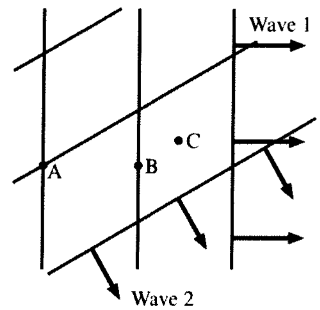
	x_1	x_2	Δx	ϕ_1	ϕ_2	$\Delta\phi$
Point A	1 m	3 m	2 m	0.5π	2.5π	2π
Point B	2 m	4 m	2 m	π	3π	2π
Point C	3 m	5 m	2 m	1.5π	3.5π	2π
Point D	4 m	6 m	2 m	2π	4π	2π

12. Review your answers to the Exercises 10 and 11. Is it the separation path length difference Δx or the phase difference $\Delta\phi$ between the waves that determines whether the interference is constructive or destructive? Explain.

The phase difference $\Delta\phi$ determines whether the interference is constructive or destructive.
Constructive: $\Delta\phi = m(2\pi)$ where $m = 0, 1, 2, 3, \ldots$
Destructive: $\Delta\phi = (m + \frac{1}{2})(2\pi)$
The separation path length difference (ΔX) alone is not sufficient to reveal the interference.
Note: $\Delta\phi = 2\pi \frac{\Delta X}{\lambda} + \Delta\phi_0$

17.7 Interference in Two and Three Dimensions

13. This is a snapshot graph of two plane waves passing through a region of space. Each has a 2 mm amplitude. At each lettered point, what are the displacements of each wave and the net displacement?

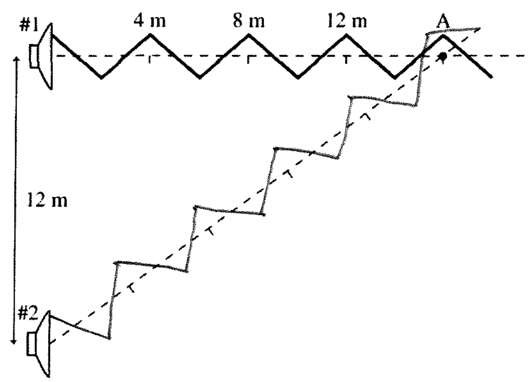


a. Point A: $D_1 =$ 2 mm $D_2 =$ 2 mm $D_{net} =$ 4 mm

b. Point B: $D_1 =$ 2 mm $D_2 =$ −2 mm $D_{net} =$ 0

c. Point C: $D_1 =$ −2 mm $D_2 =$ −2 mm $D_{net} =$ −4 mm

14. Speakers 1 and 2 are 12 m apart. Both emit identical triangular sound waves with $\lambda =$ 4 m and $\phi_0 =$ $\pi/2$ rad. Point A is $r_1 =$ 16 m from speaker 1.



a. What is distance r_2 from speaker 2 to A?

$\sqrt{(12m)^2+(16m)^2} = 20m$

b. Draw the wave from speaker 2 along the dashed line to just past point A.

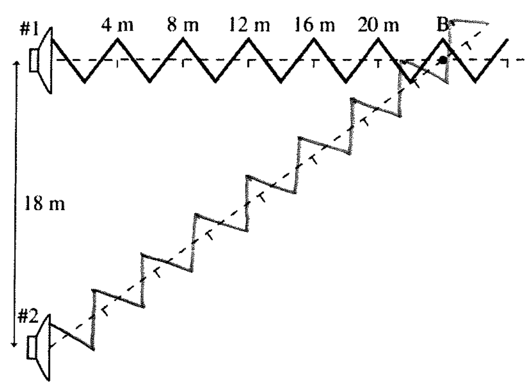
c. At A, is wave 1 a crest, trough, or zero? crest

At A, is wave 2 a crest, trough, or zero? crest

d. What is the path length difference $\Delta r = r_2 - r_1$? 4m What is the ratio $\Delta r/\lambda$? 1

e. Is the interference at point A constructive, destructive, or in between? Constructive

15. Speakers 1 and 2 are 18 m apart. Both emit identical triangular sound waves with $\lambda =$ 4 m and $\phi_0 =$ $\pi/2$ rad. Point B is $r_1 =$ 24 m from speaker 1.



a. What is distance r_2 from speaker 2 to B?

$\sqrt{(18m)^2+(24m)^2} = 30m$

b. Draw the wave from speaker 2 along the dashed line to just past point ~~A~~. B.

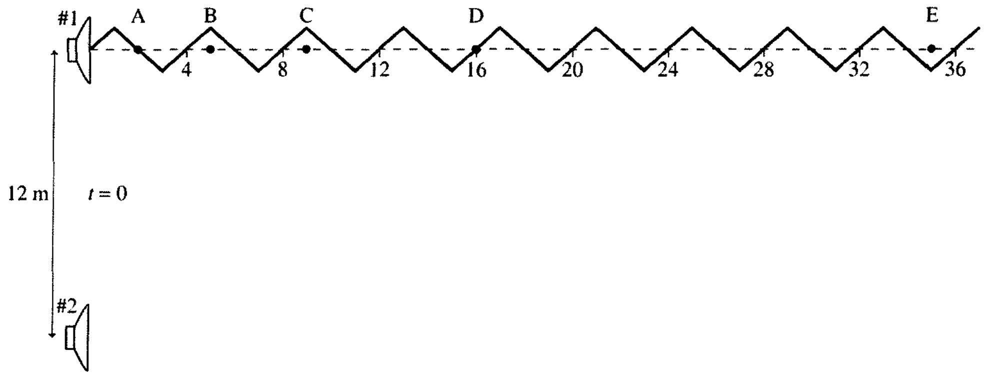
c. At B, is wave 1 a crest, trough, or zero? crest

At B, is wave 2 a crest, trough, or zero? trough

d. What is the path length difference $\Delta r = r_2 - r_1$? 6m What is the ratio $\Delta r/\lambda$? $1\frac{1}{2}$

e. Is the interference at point B constructive, destructive, or in between? Destructive

16. Two speakers 12 m apart emit identical triangular sound waves with $\lambda = 4$ m and $\phi_0 = 0$ rad. The distances r_1 to points A, B, C, D, and E are shown in the table below.



a. For each point, fill in the table and determine whether the interference is constructive (C) or destructive (D).

Point	r_1	r_2	Δr	$\Delta r/\lambda$	C or D
A	2.2 m	12.2 m	10 m	2.5	D
B	5.0 m	13 m	8 m	2	C
C	9.0 m	15 m	6 m	1.5	D
D	16 m	20 m	4 m	1	C
E	35 m	37 m	2 m	0.5	D

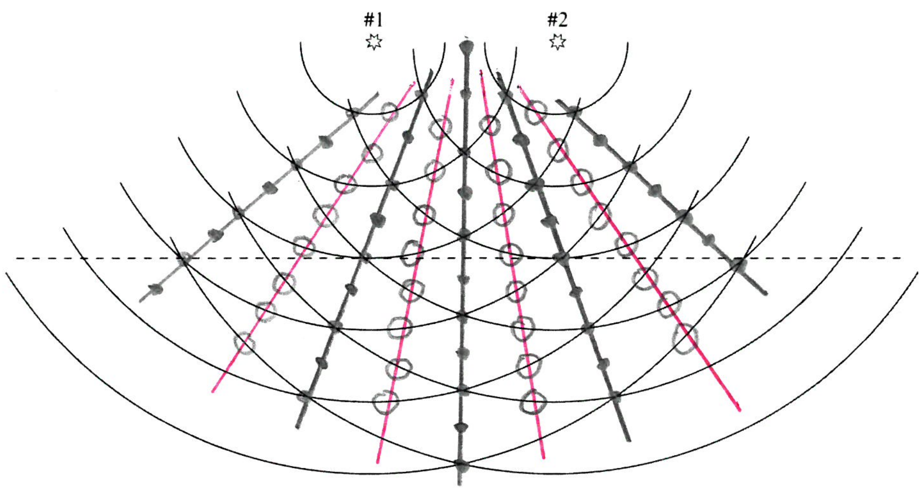
b. Are there any points to the right of E, on the line straight out from speaker 1, for which the interference is either exactly constructive or exactly destructive? If so, where? If not, why not?

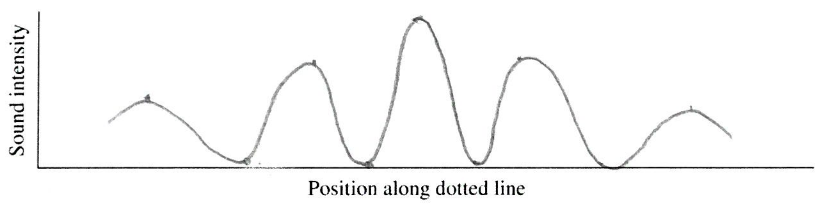
No, for all points to the right of E, the path length difference (Δr) will be less than 2m, but can never be exactly 0m. Therefore, exactly constructive or destructive interference is not possible beyond E.

c. Suppose you start at speaker 1 and walk straight away from the speaker for 50 m. Describe what you will hear as you walk.

As you walk along you will hear the sound alternate between loud and quiet (corresponding to locations of total constructive and total destructive interference). At most locations along your walk you will hear at least some sound (partial constructive interference).

17. The figure shows the wave-front pattern emitted by two loudspeakers.
    a. Draw a dot • at points where there is constructive interference. These will be points where two crests overlap *or* two troughs overlap.
    b. Draw an open circle ∘ at points where there is destructive interference. These will be points where a crest overlaps a trough.





c. Use a **black** line to draw each antinodal line of constructive interference. Use a **red** line to draw each nodal line of destructive interference.

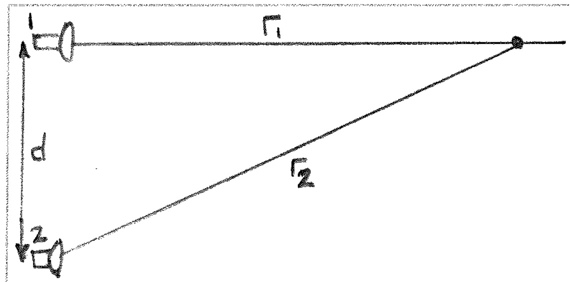
d. Draw a graph on the axes above of the sound intensity you would hear if you walked along the horizontal dashed line. Use the same horizontal scale as the figure so that your graph lines up with the figure above it.

e. Suppose the phase constant of speaker 2 is increased by π rad. Describe what will happen to the interference pattern.

The lines of constructive and destructive interference will exchange places with each other. (The pattern shifts (rotates) as the difference in phase constants ($\Delta\phi_0$) changes.)

18. Two identical, in-phase loudspeakers are in a plane, distance d apart. Both emit sound waves with wavelength λ, with $\lambda \ll d$. At what distances r_1 directly in front of speaker 1 is there maximum constructive interference of the two sound waves?
PSS 17.1

a. Begin with a visual representation. Draw two speakers, one directly above the other, on the left edge of the empty space. Label them 1 and 2, and show the distance between them as d. Draw a horizontal line straight out from speaker 1. Place a dot on this line, and label its distance from the speaker as r_1. Under what conditions is the interference at this point constructive?



b. Referring to PSS 17.1, write the condition for constructive interference for two in-phase sources. Then rearrange it to be of the form $\Delta r = \ldots$

$$2\pi \frac{\Delta r}{\lambda} + \underbrace{\cancel{\Delta\phi_0}}_{0} = m(2\pi) \quad \text{so} \quad \Delta r = m\lambda$$

where $m = 0, 1, 2, 3, \ldots$

c. Draw and label r_2 on your diagram. Write an expression for r_2 in terms of r_1 and d.

$$r_1^2 + d^2 = r_2^2 \quad \text{so} \quad r_2 = \sqrt{r_1^2 + d^2}$$

d. The path-length difference is $\Delta r = r_2 - r_1$. Use your part c result to rewrite the condition, from part b, for maximum constructive interference.

$$\Delta r = \sqrt{r_1^2 + d^2} - r_1 = m\lambda$$

e. Solve this equation for r_1. Do so by first isolating the square-root term on one side, then squaring both sides.

$$\sqrt{r_1^2 + d^2} = r_1 + m\lambda$$
$$r_1^2 + d^2 = (r_1 + m\lambda)^2 = r_1^2 + 2m\lambda r_1 + m^2\lambda^2$$
$$\frac{d^2 - (m\lambda)^2}{2m\lambda} = r_1$$

f. Is there a solution for $m = 0$? No Does this make sense? $m = 0$ corresponds to zero path-length difference. Is there any point on the line you drew where the path-length difference would be zero? Explain.

There is no value for r_1 on the drawn line that can be found with $m = 0$. Since $m = 0$ implies $\Delta r = r_2 - r_1 = 0$, the only solutions for r_1 would require either $d = 0$ (which is not possible since the two speakers can't occupy the same space) or $r_1 = \infty$.

g. Write explicit results for the values of r_1 giving constructive interference with $m = 1$ and $m = 2$. Which of these is closer to speaker 1? Explain why this is so.

For $m=1$, $r_1 = \frac{d^2-\lambda^2}{2\lambda} \approx \frac{d^2}{2\lambda}$ since $d >> \lambda$

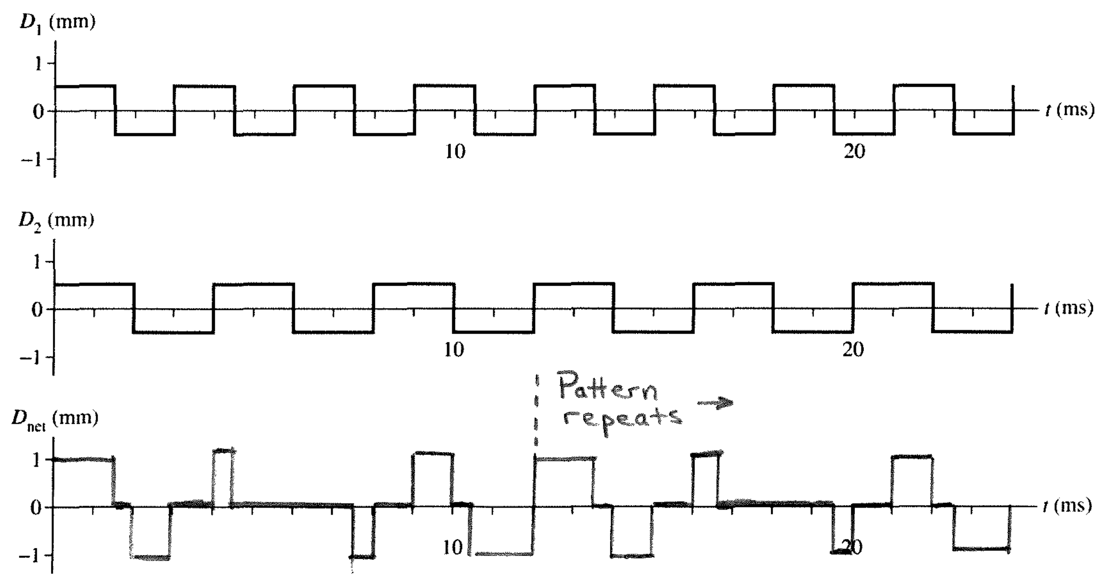
For $m=2$, $r_1 = \frac{d^2-4\lambda^2}{4\lambda} \approx \frac{d^2}{4\lambda}$ a shorter r_1 (closer to speaker 1)

h. What is the maximum value of *m* for which there is a meaningful solution? What's going on physically that prevents *m* from exceeding this value?

As m gets bigger, r_1 get smaller. The smallest value for r_1 is 0 (at the speaker) which would give a maximum value for m. If $r_1 = 0$, then the numerator $(d^2-(m\lambda)^2)$ in the solution for r_1 is 0. So, $d = m\lambda$ gives the $\boxed{\text{max } m = d/\lambda}$

17.8 Beats

19. The two waves arrive simultaneously at a point in space from two different sources.



a. Period of wave 1? 3 ms Frequency of wave 1? 333 Hz

b. Period of wave 2? 4 ms Frequency of wave 2? 250 Hz

c. Draw the graph of the net wave at this point on the third set of axes. Be accurate, use a ruler!

d. Period of the net wave? 12 ms Frequency of the net wave? 83 Hz

e. Is the frequency of the superposition what you would expect as a beat frequency? Explain.

Yes, the superposition has a (beat) frequency equal to the difference in the frequencies $(f_2 - f_1)$ of the two waves.