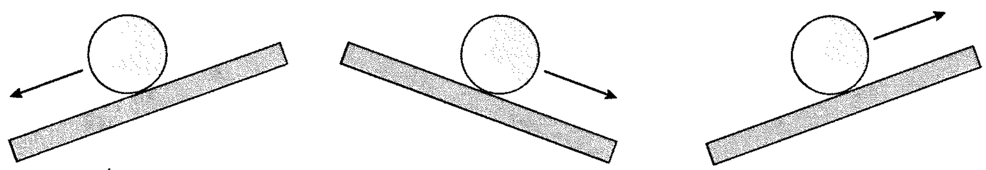
12 Rotation of a Rigid Body

12.1 Rotational Motion

1. Can the following be reasonably modeled as rigid bodies? Answer Yes or No.
   a. A car wheel and tire. Yes.
   b. A person. No.
   c. A yo-yo. Yes.
   d. A bowl of Jello. No.

2. The following figures show a wheel rolling on a ramp. Determine the signs (+ or −) of the wheel's angular velocity and angular acceleration.



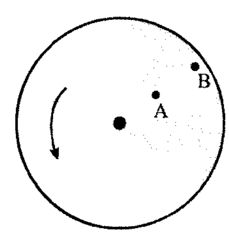
ω + $\qquad$ ω − $\qquad$ ω −

α + $\qquad$ α − $\qquad$ α +

3. A ball is rolling back and forth inside a bowl. The figure shows the ball at extreme left edge of the ball's motion as it changes direction.

   a. At this point, is ω positive, negative, or zero? 0
   b. At this point, is α positive, negative, or zero? −

4. Point B on a rotating wheel is twice as far from the axle as point A.



   a. Is ω_B equal to $\frac{1}{2}\omega_A$, ω_A, or $2\omega_A$? Explain.

$\omega_B = \omega_A = \frac{d\theta}{dt}$

All points on a rotating wheel (except the center) have the same spin rate.

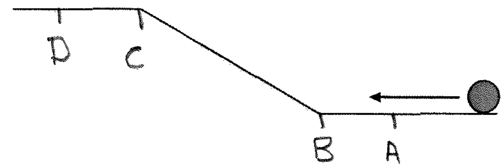
   b. Is v_B equal to $\frac{1}{2}v_A$, v_A, or $2v_A$? Explain.

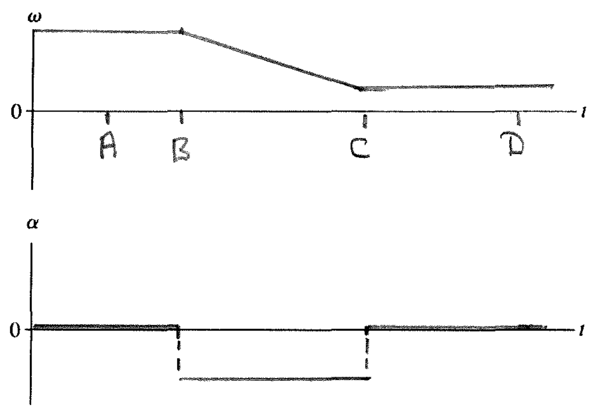
$V_B = 2V_A$

$V = r\omega$ where $\omega_B = \omega_A$ but $r_B = 2r_A$

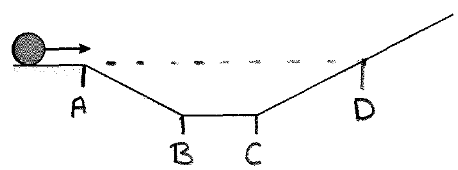
Points A and B complete one revolution over the same time interval, but B has twice the distance to travel.

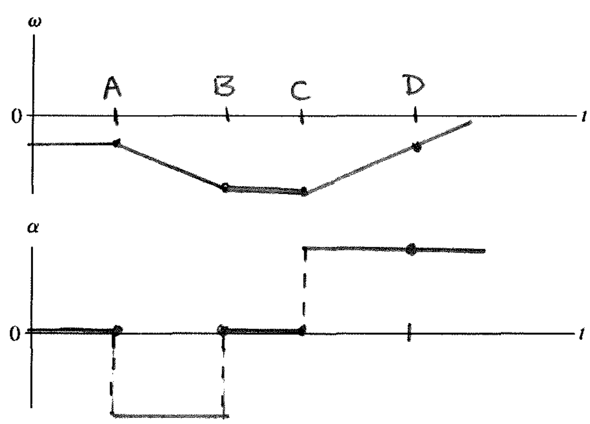
5. A wheel rolls to the left along a horizontal surface, up a ramp, then continues along the upper horizontal surface. Draw graphs for the wheel's angular velocity ω and angular acceleration α a as a function of time.





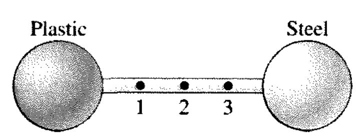
6. A wheel rolls to the right along the surface shown. Draw graphs for the wheel's angular velocity ω and angular acceleration α until the wheel reaches its highest point on the right side.





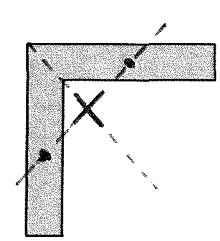
12.2 Rotation about the Center of Mass

7. Is the center of mass of this dumbbell at point 1, 2, or 3? Explain.



Point 3. The spheres are the same size, but the steel sphere is more dense and so has a greater mass.

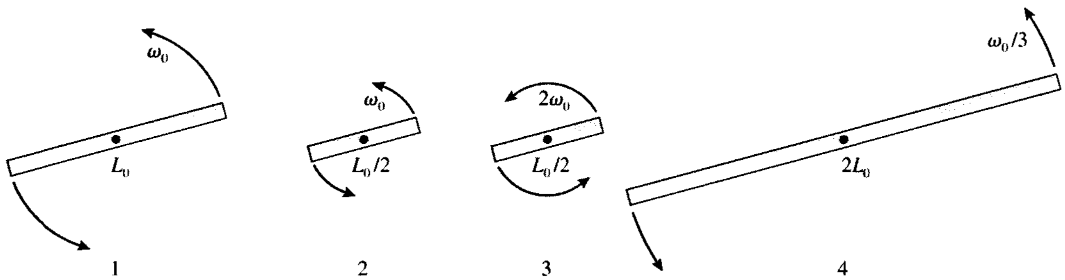
8. Mark the center of mass of this object with an ×.



12.3 Rotational Energy

12.4 Calculating Moment of Inertia

9. The figure shows four equal-mass bars rotating about their center. Rank in order, from largest to smallest, their rotational kinetic energies K_1 to K_4.



Order: $K_1 = K_3 > K_4 > K_2$

Explanation:

$K = \frac{1}{2} I \omega^2$ where $I = \frac{1}{3} m L^2$ here so $K \propto L^2 \omega^2$

$K_1 \propto L_0^2 \omega_0^2$; $K_2 \propto \left(\frac{L_0}{2}\right)^2 \omega_0^2 = \frac{K_1}{4}$; $K_3 = \left(\frac{L_0}{2}\right)^2 (2\omega_0)^2 = K_1$

$K_4 \propto (2L_0)^2 \left(\frac{\omega_0}{3}\right)^2 = \frac{4}{9} K_1$

10. Two solid spheres have the same mass. Sphere B has twice the rotational kinetic energy of sphere A.

a. What is the ratio R_B/R_A of their radii?

Since $K = \frac{1}{2} I \omega^2$ and $I \propto mR^2$ we have $K \propto R^2$

So $\frac{R_B}{R_A} = \sqrt{\frac{K_B}{K_A}} = \sqrt{2}$

b. Would your answer change if both spheres were hollow? Explain.

No. As long as both spheres are the same type (i.e., both hollow or both solid) the prefactors cancel.

c. Would your answer change if A were solid and B were hollow? Explain.

Yes. $\underset{\text{solid}}{I_A} = \frac{2}{5} m R_A^2$ but $\underset{\text{hollow}}{I_B} = \frac{2}{3} m R_B^2$

The prefactors do not cancel.

11. Which has more kinetic energy: a particle of mass M rotating with angular velocity ω in a circle of radius R, or a sphere of mass M and radius R spinning at angular velocity ω? Explain.

Since the particle has a larger moment of inertia it has more kinetic energy.

$K_{\text{particle}} = \frac{1}{2}(mR^2)(\omega^2) > K_{\text{sphere, hollow}} = \frac{1}{2}\left(\frac{2}{3} m R^2\right)\omega^2$

If the sphere is solid, replace $\frac{2}{3}$ with $\frac{2}{5}$.

12. The moment of inertia of a uniform rod about an axis through its center is $\frac{1}{12}ML^2$. The moment of inertia about an axis at one end is $\frac{1}{3}ML^2$. Explain *why* the moment of inertia is larger about the end than about the center.

The resistance to starting the spin of a rod is greater when the mass of that rod is concentrated farther away from the axis of rotation. Therefore, the rod with the axis at one of its ends has a larger moment of inertia than the rod with the axis through its center.

13. You have two steel spheres. Sphere 2 has three times the radius of sphere 1. By what *factor* does the moment of inertia I_2 of sphere 2 exceed the moment of inertia I_1 of sphere 1?

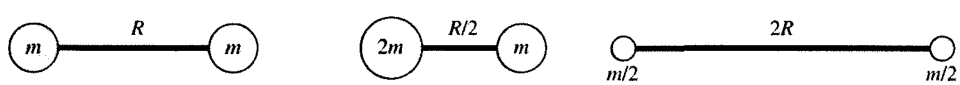
$$\frac{I_2}{I_1} = \frac{\frac{2}{5} m_2 R_2^2}{\frac{2}{5} m_1 R_1^2} = \frac{\rho_{steel}\left(\frac{4}{3}\pi R_2^3\right) R_2^2}{\rho_{steel}\left(\frac{4}{3}\pi R_1^3\right) R_1^2} = \frac{R_2^5}{R_1^5} = \frac{(3R_1)^5}{R_1^5} = 243$$

$$I_2 = 243\, I_1$$

14. The professor hands you two spheres. They have the same mass, the same radius, and the same exterior surface. The professor claims that one is a solid sphere and that the other is hollow. Can you determine which is which without cutting them open? If so, how? If not, why not?

It will be easier to start the rotation of the solid sphere because the hollow sphere's mass is generally distributed farther away from its center. The solid sphere will roll down an incline at a faster rate.

15. Rank in order, from largest to smallest, the moments of inertia I_1, I_2, and I_3 about the midpoint of the rod.



1 2 3

Order: $I_3 > I_1 > I_2$ Assume massless rod.

Explanation: Assume rotation axis is perpendicular to rod.

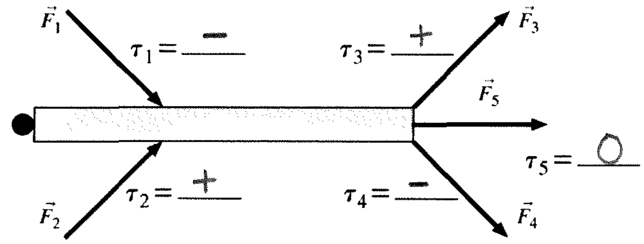
$$I_1 \approx 2m\left(\frac{R}{2}\right)^2 = \frac{mR^2}{2}$$

$$I_2 \approx 2m\left(\frac{R}{4}\right)^2 + m\left(\frac{R}{4}\right)^2 = \frac{3mR^2}{16}$$

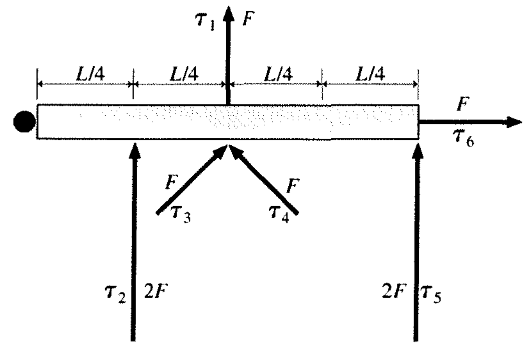
$$I_3 \approx \frac{m}{2}R^2 + \frac{m}{2}R^2 = mR^2$$

12.5 Torque

16. Five forces are applied to a door. For each, determine if the torque about the hinge is positive (+), negative (−), or zero (0).



17. Six forces, each of magnitude either F or $2F$, are applied to a door. Rank in order, from largest to smallest, the six torques τ_1 to τ_6 about the hinge.



Order: $\tau_5 > \tau_1 = \tau_2 > \tau_3 = \tau_4 > \tau_6$

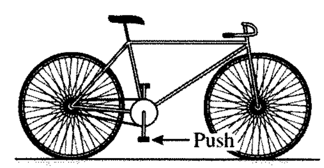
Explanation:

$\tau_5 = L(2F)$ $\quad \tau_3 = \frac{L}{2} F \sin 45^\circ$

$\tau_1 = \frac{L}{2}(F)$ $\quad \tau_4 = \frac{L}{2} F \sin 45^\circ$

$\tau_2 = \frac{L}{4}(2F)$ $\quad \tau_6 = LF \sin 0^\circ = 0$

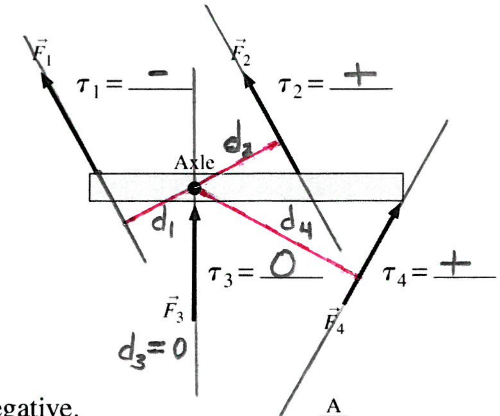
18. A bicycle is at rest on a smooth surface. A force is applied to the bottom pedal as shown. Does the bicycle roll forward (to the right), backward (to the left), or not at all? Explain.



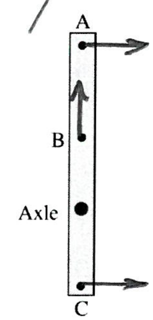
The bicycle will not roll at all.
Though the chain exerts a torque on the rear wheel that will rotate it clockwise (as if to move the bicycle forward), the bicycle does not move forward because the surface is smooth.
The tire needs to push against the surface, and the surface push back on the tire, to move the bicycle forward (to the right).

19. Four forces are applied to a rod that can pivot on an axle. For each force,

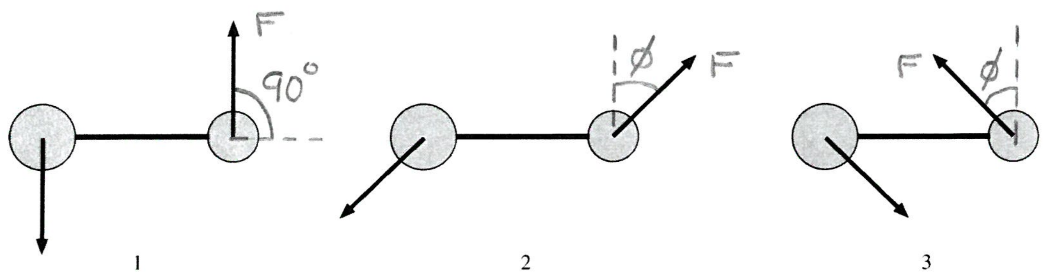
   a. Use a **black** pen or pencil to draw the line of action.
   b. Use a **red** pen or pencil to draw and label the moment arm, or state that $d = 0$.
   c. Determine if the torque about the axle is positive (+), negative (−) or zero (0). Write your answer in the blank.



20. a. Draw a force vector at A whose torque about the axle is negative.
   b. Draw a force vector at B whose torque about the axle is zero.
   c. Draw a force vector at C whose torque about the axle is positive.



21. The dumbbells below are all the same size, and the forces all have the same magnitude. Rank in order, from largest to smallest, the torques τ_1, τ_2, and τ_3 about the midpoint of the rod.



Order: $\tau_1 > \tau_2 = \tau_3$

Explanation: Same moment arms

Net torque $\tau_1 \propto 2F \sin 90° = 2F$

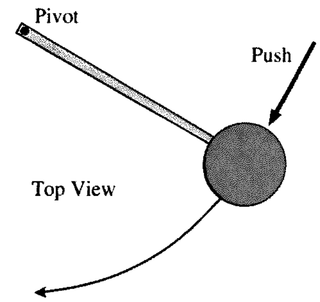
Net torque $\tau_2 \propto 2F \sin(90° - \phi) < 2F$

Net torque $\tau_3 \propto 2F \sin(90° + \phi) < 2F$

But $\sin(90° - \phi) = \sin(90° + \phi)$

12.6 Rotational Dynamics

22. A student gives a quick push to a ball at the end of a massless, rigid rod, causing the ball to rotate clockwise in a *horizontal* circle. The rod's pivot is frictionless.



a. As the student is pushing, is the torque about the pivot positive, negative, or zero?

negative

b. After the push has ended, does the ball's angular velocity

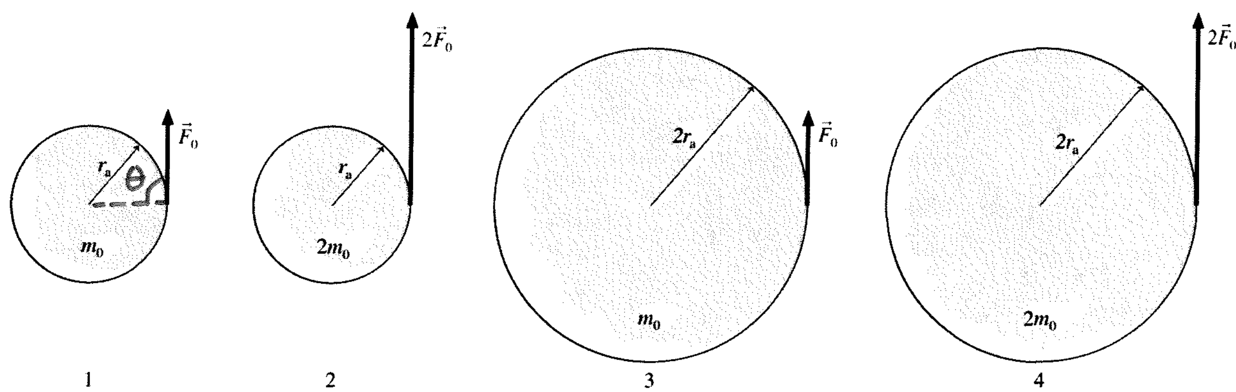
i. Steadily increase?
ii. Increase for awhile, then hold steady?
iii. Hold steady?
iv. Decrease for awhile, then hold steady?
v. Steadily decrease?

Explain the reason for your choice.

Since the rotation of the system is on a horizontal plane, after the push ends the net torque on the system is zero, and no angular acceleration occurs.

c. Right after the push has ended, is the torque positive, negative, or zero? zero.

23. a. Rank in order, from largest to smallest, the torques τ_1 to τ_4 about the center of the wheel.



Order: $\tau_4 > \tau_2 = \tau_3 > \tau_1$

Explanation: $\tau = rF\sin\theta = rF\sin 90^\circ = rF$

$\tau_4 = (2r_0)(2F_0)$, $\tau_2 = r_0(2F_0)$, $\tau_3 = (2r_0)F_0$, $\tau_1 = r_0 F_0$

b. Rank in order, from largest to smallest, the angular accelerations α_1 to α_4.

In (a) above, each τ = net torque since only one F given.

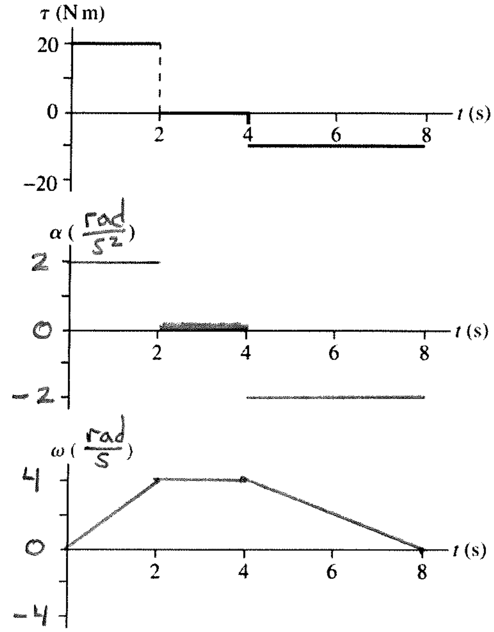
$\alpha = \dfrac{\tau_{net}}{I}$, $\boxed{\alpha_1 = \alpha_2 > \alpha_3 = \alpha_4}$ Use $I = mR^2$ since all m at rim of wheel.

$\alpha_1 = \dfrac{r_0 F_0}{m_0 r_0^2}$, $\alpha_2 = \dfrac{2r_0F_0}{2m_0r_0^2}$, $\alpha_3 = \dfrac{2r_0F_0}{m_0(2r_0)^2}$, $\alpha_4 = \dfrac{4r_0F_0}{2m_0(2r_0)^2}$

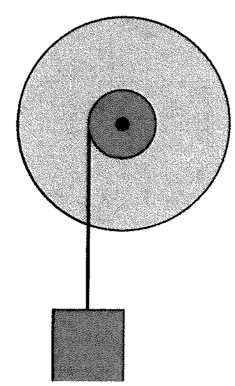
24. The top graph shows the torque on a rotating wheel as a function of time. The wheel's moment of inertia is $10\ \text{kg m}^2$. Draw graphs of α-versus-t and ω-versus-t, assuming $\omega_0 = 0$. Provide units and appropriate scales on the vertical axes.

$\tau_{net} = I\alpha$, $\alpha = \dfrac{\tau_{net}}{I}$

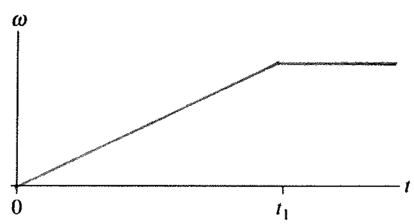
$\omega = \omega_0 + \alpha t$



25. The wheel turns on a frictionless axle. A string wrapped around the smaller diameter shaft is tied to a block. The block is released at $t = 0$ s and hits the ground at $t = t_1$.



a. Draw a graph of ω-versus-t for the wheel, starting at $t = 0$ s and continuing to some time $t > t_1$.

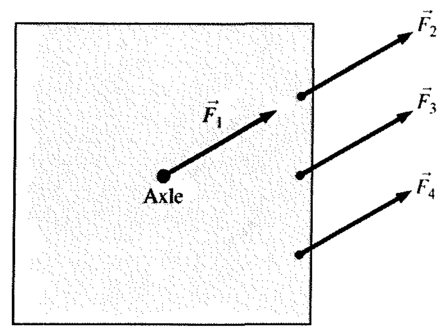


b. Is the magnitude of the block's downward acceleration greater than g, less than g, or equal to g? Explain.

Tension in the rope is needed to create a torque that gives angular acceleration to the shaft and wheel. The block's downward acceleration is less than g since the rope tension is not zero.

12.7 Rotation about a Fixed Axis

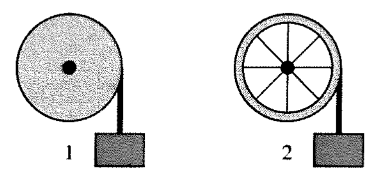
26. A square plate can rotate about an axle through its center. Four forces of equal magnitude are applied, one at a time, to different points on the plate. The forces turn as the plate rotates, maintaining the same orientation with respect to the plate. Rank in order, from largest to smallest, the angular accelerations α_1 to α_4 caused by the four forces.



Order: $\alpha_4 > \alpha_3 > \alpha_2 = \alpha_1$
Explanation:

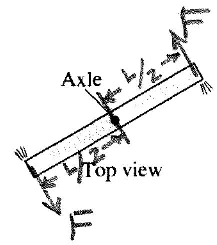
Both $\vec{F}_2$ and $\vec{F}_1$ act along a line through the axle giving no moment arm ($d_1 = d_2 = 0$). $\vec{F}_4$ has the longest moment arm.

27. A solid cylinder and a cylindrical shell have the same mass, same radius, and turn on frictionless, horizontal axles. (The cylindrical shell has light-weight spokes connecting the shell to the axle.) A rope is wrapped around each cylinder and tied to a block. The blocks have the same mass and are held the same height above the ground. Both blocks are released simultaneously. The ropes do not slip.



Which block hits the ground first? Or is it a tie? Explain.

The block attached to the solid cylinder hits first. Both cylinders experience the same unbalanced torque, but the solid cylinder has smaller moment of inertia so it has less resistance to angular acceleration ($\alpha = \frac{\tau_{net}}{I}$).



28. A metal bar of mass M and length L can rotate in a horizontal plane about a vertical, frictionless axle through its center. A hollow channel down the bar allows compressed air (fed in at the axle) to spray out of two small holes at the ends of the bar, as shown. The bar is found to speed up to angular velocity ω in a time interval Δt, starting from rest. What force does each escaping jet of air exert on the bar?
PSS 12.1

a. On the figure, draw vectors to show the forces exerted on the bar. Then label the moment arms of each force.

b. The forces in your drawing exert a torque about the axles. Write an expression for each torque, and then add them to get the net torque. Your expression should be in terms of the unknown force F and "known" quantities such as M, L, g, etc.

Net $\tau = F\left(\frac{L}{2}\right) + F\left(\frac{L}{2}\right) = FL$

Both torques give counterclockwise rotation.

c. What is the moment of inertia of this bar about the axle? $\frac{1}{12} m L^2$

d. According to Newton's second law, the torque causes the bar to undergo an angular acceleration. Use your results from parts b and c to write an expression for the angular acceleration. Simplify the expression as much as possible.

Net $\tau = I\alpha$ or $\alpha = \frac{\tau_{net}}{I} = \frac{FL}{\frac{1}{12} m L^2} = \frac{12F}{mL}$

e. You can now use rotational kinematics to write an expression for the bar's angular velocity after time Δt has elapsed. Do so.

$\omega = \omega_0 + \alpha \Delta t = \alpha \Delta t = \frac{12F}{mL} \Delta t$

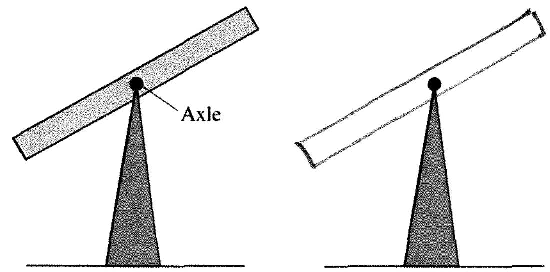
f. Finally, solve your equation in part e for the unknown force.

$\frac{mL\omega}{12 \Delta t} = F$

This is now a result you could use with experimental measurements to determine the size of the force exerted by the gas.

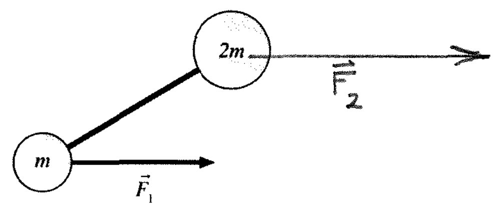
12.8 Static Equilibrium

29. A uniform rod pivots about a frictionless, horizontal axle through its center. It is placed on a stand, held motionless in the position shown, then gently released. On the right side of the figure, draw the final, equilibrium position of the rod. Explain your reasoning.



The rod remains at rest and does not rotate since there is no unbalanced torque acting on the rod. The rod's weight acts at the center of gravity for rod which is also the pivot so the moment arm for this force is zero.

30. The dumbbell has masses m and $2m$. Force $\vec{F}_1$ acts on mass m in the direction shown. Is there a force $\vec{F}_2$ that can act on mass $2m$ such that the dumbbell moves with pure translational motion, without any rotation? If so, draw $\vec{F}_2$, making sure that its length shows the magnitude of $\vec{F}_2$ relative to $\vec{F}_1$. If not, explain why not.

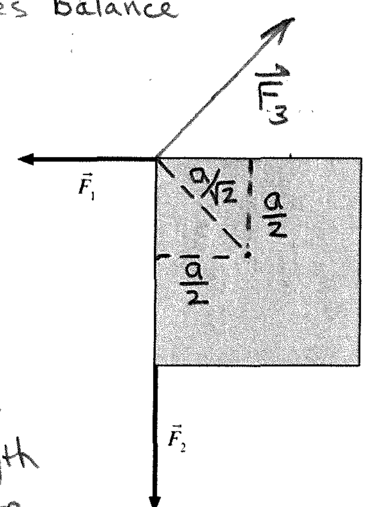


As drawn, $\vec{F}_2 = 2\vec{F}_1$. The center of mass of the dumbell is $\frac{2}{3}$ of the way from m to 2m. This gives a moment arm for $\vec{F}_1$ that is twice that of $\vec{F}_2$, that is, $d_1 = 2d_2$. The torques balance out:

$\vec{\tau}_1 = d_1 F_1$ C.C.W.

$\vec{\tau}_2 = \left(\frac{d_1}{2}\right)(2F_1)$ C.W.

31. Forces $\vec{F}_1$ and $\vec{F}_2$ have the same magnitude and are applied to the corners of a square plate. Is there a *single* force $\vec{F}_3$ that, if applied to the appropriate point on the plate, will cause the plate to be in total equilibrium? If so, draw it, making sure it has the right position, orientation, and length. If not, explain why not.



As drawn, $\vec{F}_3 = -(\vec{F}_1 + \vec{F}_2)$ so the forces sum to zero. Let a = length of the side of the given square plate, then notice the torques sum to zero too.

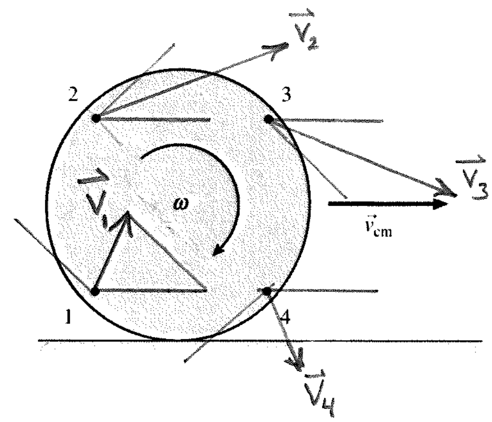
$F_3 = \sqrt{2}\,F$ where $F = F_1 = F_2$

$d_3 = \frac{a}{\sqrt{2}}$ so $\tau_3 = \left(\frac{a}{\sqrt{2}}\right)(\sqrt{2}\,F) = aF$ C.W.

and $\tau_1 + \tau_2 = \left(\frac{a}{2}\right)F + \left(\frac{a}{2}\right)F = aF$ C.C.W.

12.9 Rolling Motion

32. A wheel is rolling along a horizontal surface with the center-of-mass velocity shown. Draw the velocity vector $\vec{v}$ at points 1 to 4 on the rim of the wheel.



Notice

$|\vec{V}_1| = |\vec{V}_4|$

$|\vec{V}_2| = |\vec{V}_3|$

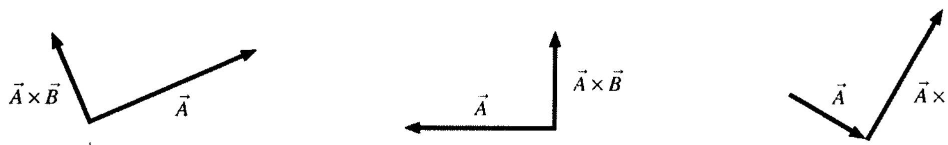
12.10 The Vector Description of Rotational Motion

12.11 Angular Momentum

33. For each vector pair $\vec{A}$ and $\vec{B}$ shown below, determine if $\vec{A} \times \vec{B}$ points into the page, out of the page, or is zero.

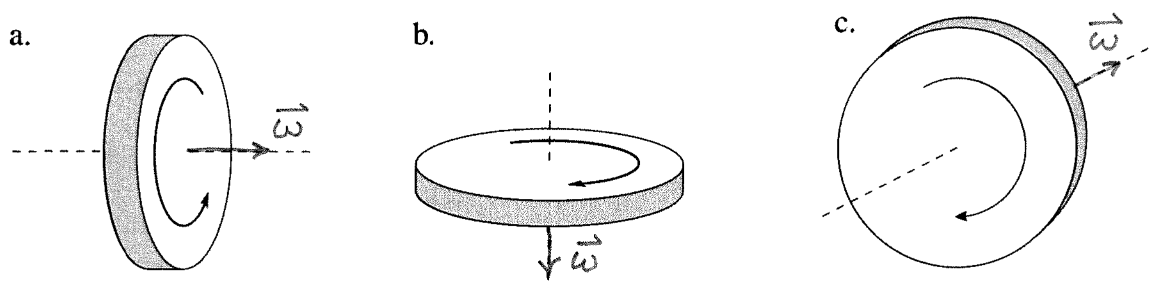
$\vec{A} \times \vec{B}$ into page $\vec{A} \times \vec{B}$ out of page $\vec{A} \times \vec{B}$ out of page $\vec{A} \times \vec{B}$ 0

34. Each figure below shows $\vec{A}$ and $\vec{A} \times \vec{B}$. Determine if $\vec{B}$ is in the plane of the page or perpendicular to the page. If $\vec{B}$ is in the plane of the page, draw it. If $\vec{B}$ is perpendicular to the page, state whether $\vec{B}$ points into the page or out of the page.

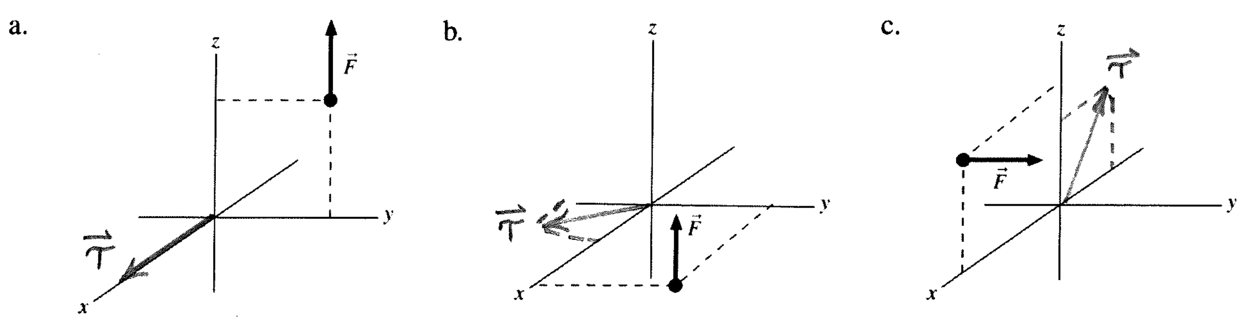


$\vec{B}$ into page $\vec{B}$ out of page $\vec{B}$ into page

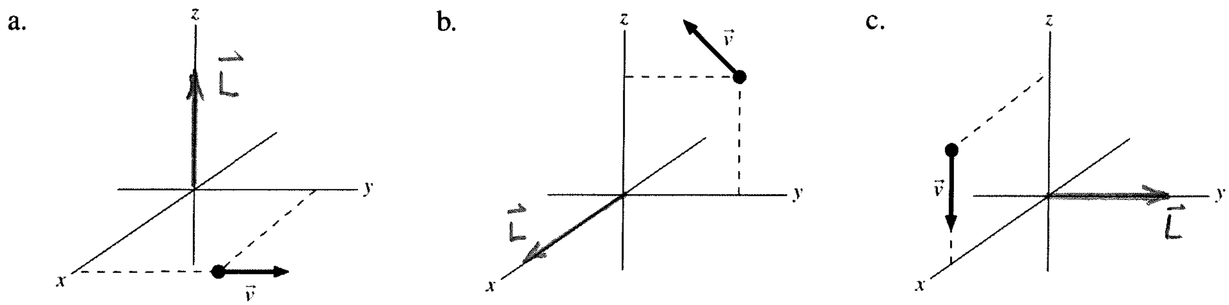
35. Draw the angular velocity vector on each of the rotating wheels.



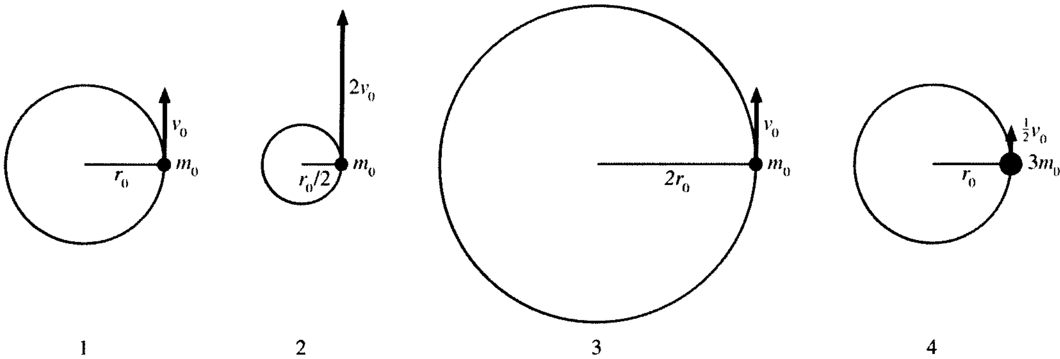
36. The figures below show a force acting on a particle. For each, draw the torque vector for the torque about the origin.
    - Place the tail of the torque vector at the origin.
    - Draw the vector large and straight (use a ruler!) so that its direction is clear. Use dotted lines from the tip of the vector to the axes to show the plane in which the vector lies.



37. The figures below show a particle with velocity $\vec{v}$. For each, draw the angular momentum vector $\vec{L}$ for the angular momentum relative to the origin. Place the tail of the angular momentum vector at the origin.



38. Rank in order, from largest to smallest, the angular momenta L_1 to L_4.



Order: $L_3 > L_4 > L_2 = L_1$

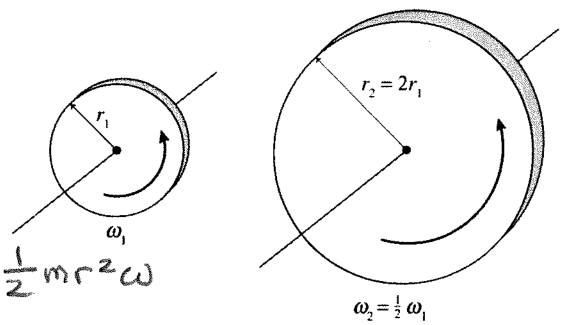
Explanation:

$\vec{L} = \vec{r} \times \vec{p}$

$L_1 = r_0 m_0 v_0$, $L_2 = \frac{r_0}{2} m_0 (2v_0)$

$L_3 = 2r_0 m_0 v_0$, $L_4 = r_0 3m_0 \frac{v_0}{2}$

39. Disks 1 and 2 have equal mass. Is the angular momentum of disk 2 larger than, smaller than, or equal to the angular momentum of disk 1? Explain.



$L_2 > L_1$ (larger than)

Let $m = m_1 = m_2$. $L = I\omega = \frac{1}{2}mr^2\omega$

$L_1 = \frac{1}{2} m r_1^2 \omega_1^2$

$L_2 = \frac{1}{2} m (2r_1)^2 (\frac{1}{2}\omega_1) = m r_1^2 \omega_1^2 = 2L_1$